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***q*-Deformed Fock spaces and modular representations of spin symmetric groups**

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Abstract. We use the Fock-space representation of the quantum affine algebra of type $A_{2n}^{(2)}$ to obtain a description of the global crystal basis of its basic level 1 module. We formulate a conjecture relating this basis to decomposition matrices of spin symmetric groups in characteristic $2n + 1$.

1. Introduction

The Fock-space representation of the quantum affine algebra $U_q(\widehat{sl}_n) = U_q(A_{n-1}^{(1)})$ was constructed by Hayashi [16]. A combinatorial version of this construction was then used by Misra and Miwa [29] to describe Kashiwara's crystal basis of the basic representation $V(\Lambda_0)$. This made it possible to compute the global crystal basis of $V(\Lambda_0)$ [26]. Then, it was conjectured that the degree m part of the transition matrices giving the coefficients of the global basis on the natural basis of the Fock space were q analogues of the decomposition matrices of the type A Hecke algebras H_m at an n th root of unity [26]. According to a conjecture of James [17], these should coincide, for n prime and large enough, with the decomposition matrices of symmetric groups S_m over a field of characteristic n . The conjecture of [26] has been proved by Ariki [3], and by Grojnowski [15] using the results of [14].

There is another approach to the calculation of decomposition matrices of type A Hecke algebras, relying upon Soergel's results on tilting modules for quantum groups at roots of unity [36, 37]. This approach also leads to q analogues of decomposition numbers expressed in terms of Kazhdan–Lusztig polynomials. It seems that these q analogues are the same as those of [26] but there is no proof of this coincidence. In fact, the relationship between the two approaches is still unclear.

The results of [26, 3, 15] have been applied recently by Foda *et al* [12] to determine which simple H_m modules remain simple after restriction to H_{m-1} and to show that this problem is equivalent to the decomposition of a tensor product of level 1 $A_{n-1}^{(1)}$ modules. This provided an explanation for an intriguing correspondence previously observed in [13] between a class of RSOS models and modular representations of symmetric groups.

Another description of the $U_q(A_{n-1}^{(1)})$ Fock space, as a deformation of the infinite wedge realization of the fermionic Fock space, was obtained by Stern [38]. In [25], the q bosons needed for the decomposition of the Fock space into irreducible $U_q(A_{n-1}^{(1)})$ modules were

introduced. This construction was used in [27] to give a combinatorial formula for the highest weight vectors, and in [28] to define a canonical basis of the whole Fock space which was conjectured to yield the decomposition matrices of q Schur algebras at roots of unity. Moreover, strong support in favour of this conjecture was obtained by establishing its compatibility with a version of the Steinberg tensor product theorem proved by James in this context [17, 28].

Recently, the theory of perfect crystals [21, 22] allowed Kashiwara *et al* [24] to define a general notion of q Fock space, extending the results of [25] to several series of affine algebras. Their results apply in particular to the twisted affine algebra of type $A_{2n}^{(2)}$, which is the case considered in this note.

It has been noticed by Nakajima and Yamada [33] that the combinatorics of the basic representation $V(\Lambda_n)$ of $A_{2n}^{(2)}$ was similar to the one encountered in the $(2n + 1)$ modular representation theory of the spin symmetric groups \widehat{S}_m by Morris [30] as early as 1965. This can be explained to a certain extent by observing that the (r, \bar{r}) -inducing operators of Morris and Yaseen [32] coincide with the Chevalley lowering operators of the Fock-space representation of $A_{2n}^{(2)}$. This provides a further example of the phenomenon observed in [26] in the case of symmetric groups and $A_{n-1}^{(1)}$ algebras.

In this note, we give the analogues for $U_q(A_{2n}^{(2)})$ of the results of [26]. Using the level 1 q Fock spaces of [24], we describe an algorithm for computing the canonical basis of the basic representation $V(\Lambda_n)$, which allows us to prove that this basis is in the $\mathbb{Z}[q]$ lattice spanned by the natural basis of the q -Fock space, and that the transition matrices have an upper triangle of zeros (theorem 4.1).

We conjecture that the specialization $q = 1$ gives, up to splitting of rows and columns for pairs of associate characters, and for sufficiently large primes $p = 2n + 1$, the decomposition matrices of spin-symmetric groups. However, the reduction $q = 1$ is more tricky than in the $A_{n-1}^{(1)}$ case. Indeed, the q -Fock space of $A_{2n}^{(2)}$ is strictly larger than the classical one, and one has to factor out the null space of a certain quadratic form [24] to recover the usual description.

The missing ingredient in the spin case when we compare it with [26] is that, since the spin-symmetric groups are not Coxeter groups, there is no standard way of associating to them a Hecke algebra, and this is an important obstruction for proving our conjecture. What we can actually prove is that all self-associate projective characters of \widehat{S}_m are linear combinations of characters obtained from smaller groups by a sequence of (r, \bar{r}) inductions (theorem 6.1). This proof is constructive in the sense that the intermediate basis $\{A(\mu)\}$ of our algorithm for the canonical basis, suitably specialized at $q = 1$, is a basis for the space spanned by such characters.

This should have implications on the way of labelling the irreducible modular spin representations of \widehat{S}_m . Up to now, a coherent labelling scheme has been found only for $p = 3$ [8] and $p = 5$ [1]. The case $p \geq 7$ led to formidable difficulties. To overcome this problem, we propose to use the labels of the crystal graph of $V(\Lambda_n)$, which may contain partitions with repeated parts not arising in the representation theory of \widehat{S}_m , and corresponding to ghost vectors of the q -Fock space at $q = 1$.

2. The Fock-space representation of $U_q(A_{2n}^{(2)})$

The Fock-space representation of the affine Lie algebra $A_{2n}^{(2)}$ can be constructed by means of its embedding in $b_\infty = \widehat{g\delta}_\infty$, the completed infinite rank affine Lie algebra of type B [10, 11].

The (bosonic) Fock space of type B is the polynomial algebra $\mathcal{F} = \mathbb{C}[p_{2j+1}, j \geq 0]$ in an infinite number of generators p_{2j+1} of odd degree $2j + 1$. If one identifies p_k with the power sum symmetric function $p_k = \sum_i x_i^k$ in some infinite set of variables, the natural basis of weight vectors for b_∞ is given by Schur's P functions P_λ (where λ runs over the set DP of partitions into distinct parts) [10, 40, 19].

The Chevalley generators e_i^∞, f_i^∞ ($i \geq 0$) of b_∞ act on P_λ by

$$e_i^\infty P_\lambda = P_\mu \quad f_i^\infty P_\lambda = P_\nu \tag{1}$$

where μ (resp. ν) is obtained from λ by replacing its part $i + 1$ by i (resp. its part i by $i + 1$), the result being 0 if $i + 1$ (resp. i) is not a part of λ . Also, it is understood that $P_\mu = 0$ as soon as μ has a multiple part. For example, $f_0^\infty P_{32} = P_{321}, f_3^\infty P_{32} = P_{42}, e_1^\infty P_{32} = P_{31}$ and $e_2^\infty P_{32} = P_{22} = 0$.

Let $h = 2n + 1$. The Chevalley generators e_i, f_i of $A_{2n}^{(2)}$ will be realized as

$$f_i = \sum_{j \equiv n \pm i} f_j^\infty \quad (i = 0, \dots, n) \tag{2}$$

$$e_i = \sum_{j \equiv n \pm i} e_j^\infty \quad (i = 0, \dots, n - 1) \quad e_n = e_0^\infty + 2 \sum_{\substack{j > 0 \\ j \equiv 0, -1}} e_j^\infty \tag{3}$$

where all congruences are taken modulo h . Let $A_{2n}^{(2)'}$ be the derived algebra of $A_{2n}^{(2)}$ (obtained by omitting the degree operator d). The action of $A_{2n}^{(2)'}$ on \mathcal{F} is centralized by the Heisenberg algebra generated by the operators $\frac{\partial}{\partial p_{hs}}$ and p_{hs} for odd $s \geq 1$. This implies that the Fock space decomposes under $A_{2n}^{(2)}$ as

$$\mathcal{F} = \bigoplus_{k \geq 0} V(\Lambda_n - k\delta)^{\oplus p^*(k)} \tag{4}$$

where $p^*(k)$ is the number of partitions of k into odd parts. In particular, the subrepresentation generated by the vacuum vector $|0\rangle = P_0 = 1$ is the basic representation $V(\Lambda_n)$ of $A_{2n}^{(2)}$, and its principally specialized character is [20]

$$\text{ch}_t V(\Lambda_n) = \sum_{m \geq 0} \dim V(\Lambda_n)_m t^m = \prod_{\substack{i \text{ odd} \\ i \not\equiv 0 \pmod{h}}} \frac{1}{1 - t^i}. \tag{5}$$

The q -deformation of this situation has been discovered by Kashiwara *et al* [24]. Contrary to the case of $A_{n-1}^{(1)}$, the q -Fock space is strictly larger than the classical one. We recall here briefly their construction, referring to [24] for details and notation.

Let $\text{DP}_h(m)$ be the set of partitions $\lambda = (1^{m_1} 2^{m_2} \dots r^{m_r})$ of m for which $m_i \leq 1$ when $i \not\equiv 0 \pmod{h}$. For example, $\text{DP}_3(7) = \{(7), (61), (52), (43), (421), (331)\}$. Set $\text{DP}_h = \bigcup_m \text{DP}_h(m)$. Then, the q -Fock space of type $A_{2n}^{(2)}$ is

$$\mathcal{F}_q = \bigoplus_{\lambda \in \text{DP}_h} \mathcal{Q}(q)|\lambda\rangle \tag{6}$$

where for $\lambda = (\lambda_1, \dots, \lambda_r)$, $|\lambda\rangle$ denotes the infinite q -wedge product

$$|\lambda\rangle = u_\lambda = u_{\lambda_1} \wedge_q u_{\lambda_2} \wedge_q \dots \wedge_q u_{\lambda_r} \wedge_q u_0 \wedge_q u_0 \wedge_q \dots$$

of basis vectors u_i of the representation V_{aff} . The quantum affine algebra $U_q(A_{2n}^{(2)})$ acts on $V_{\text{aff}} = \bigoplus_{i \in \mathbb{Z}} \mathcal{Q}(q)u_i$ by

$$f_i u_j = \begin{cases} u_{j+1} & \text{if } j \equiv n \pm i \pmod{h} \\ 0 & \text{otherwise} \end{cases} \quad (i = 0, \dots, n - 1) \tag{7}$$

$$f_n u_j = \begin{cases} u_{j+1} & \text{if } j \equiv -1 \pmod{h} \\ (q + q^{-1})u_{j+1} & \text{if } j \equiv 0 \pmod{h} \\ 0 & \text{otherwise} \end{cases} \tag{8}$$

$$e_i u_j = \begin{cases} u_{j-1} & \text{if } j \equiv n + 1 \pm i \pmod{h} \\ 0 & \text{otherwise} \end{cases} \quad (i = 0, \dots, n - 1) \tag{9}$$

$$e_n u_j = \begin{cases} u_{j-1} & \text{if } j \equiv 1 \pmod{h} \\ (q + q^{-1})u_{j-1} & \text{if } j \equiv 0 \pmod{h} \\ 0 & \text{otherwise} \end{cases} \tag{10}$$

$$t_0 u_j = \begin{cases} q^4 u_j & \text{if } j \equiv n \pmod{h} \\ q^{-4} u_j & \text{if } j \equiv n + 1 \pmod{h} \\ u_j & \text{otherwise} \end{cases} \tag{11}$$

$$t_i u_j = \begin{cases} q^2 u_j & \text{if } j \equiv n \pm i \pmod{h} \\ q^{-2} u_j & \text{if } j \equiv n + 1 \pm i \pmod{h} \\ u_j & \text{otherwise} \end{cases} \quad (i = 1, \dots, n - 1) \tag{12}$$

$$t_n u_j = \begin{cases} q^2 u_j & \text{if } j \equiv -1 \pmod{h} \\ q^{-2} u_j & \text{if } j \equiv 1 \pmod{h} \\ u_j & \text{otherwise.} \end{cases} \tag{13}$$

The only commutation rules we will need to describe the action of e_i and f_i on \mathcal{F}_q are:

$$u_j \wedge_q u_j = 0 \quad \text{if } j \not\equiv 0 \pmod{h} \tag{14}$$

$$u_j \wedge_q u_{j+1} = -q^2 u_{j+1} \wedge_q u_j \quad \text{if } j \equiv 0, -1 \pmod{h}. \tag{15}$$

The action on the vacuum vector $|0\rangle = u_0 \wedge_q u_0 \wedge_q \dots$ is given by

$$e_i |0\rangle = 0 \quad f_i |0\rangle = \delta_{in} |1\rangle \quad t_i |0\rangle = q^{\delta_{in}} |0\rangle \tag{16}$$

and on a q -wedge $|\lambda\rangle = u_{\lambda_1} \wedge_q \dots \wedge_q u_{\lambda_r} \wedge_q |0\rangle$,

$$f_i |\lambda\rangle = f_i u_{\lambda_1} \wedge_q t_i u_{\lambda_2} \wedge_q \dots \wedge_q t_i u_{\lambda_r} \wedge_q t_i |0\rangle + u_{\lambda_1} \wedge_q f_i u_{\lambda_2} \wedge_q \dots \wedge_q t_i u_{\lambda_r} \wedge_q t_i |0\rangle + \dots + u_{\lambda_1} \wedge_q u_{\lambda_2} \wedge_q \dots \wedge_q u_{\lambda_r} \wedge_q f_i |0\rangle \tag{17}$$

$$e_i |\lambda\rangle = t_i^{-1} u_{\lambda_1} \wedge_q t_i^{-1} u_{\lambda_2} \wedge_q \dots \wedge_q t_i^{-1} u_{\lambda_r} \wedge_q e_i |0\rangle + t_i^{-1} u_{\lambda_1} \wedge_q t_i^{-1} u_{\lambda_2} \wedge_q \dots \wedge_q e_i u_{\lambda_r} \wedge_q |0\rangle + \dots + e_i u_{\lambda_1} \wedge_q u_{\lambda_2} \wedge_q \dots \wedge_q u_{\lambda_r} \wedge_q |0\rangle \tag{18}$$

$$t_i |\lambda\rangle = t_i u_{\lambda_1} \wedge_q t_i u_{\lambda_2} \wedge_q \dots \wedge_q t_i u_{\lambda_r} \wedge_q t_i |0\rangle. \tag{19}$$

For example, with $n = 2$, one has

$$f_2 |542\rangle = (q^4 + q^2) |642\rangle + q |552\rangle + |5421\rangle$$

and

$$f_2 |552\rangle = (q^2 + 1) (|652\rangle + |562\rangle) + |5521\rangle = (1 - q^4) |652\rangle + |5521\rangle$$

the last equality resulting from (15).

It is proved in [24] that \mathcal{F}_q is an integrable highest weight $U_q(A_{2n}^{(2)})$ module whose decomposition into irreducible components, obtained by means of q -bosons, is

$$\mathcal{F}_q = \bigoplus_{k \geq 0} V(\Lambda_n - k\delta)^{\oplus p(k)} \tag{20}$$

where $p(k)$ is now the number of all partitions of k (compare (4)). Thus, the submodule $U_q(A_{2n}^{(2)}) |0\rangle$ is a realization of the basic representation $V(\Lambda_n)$.

3. The crystal graph of the q -Fock space

The first step in computing the global basis of $V(\Lambda_n) \subset \mathcal{F}_q$ is to determine the crystal basis of \mathcal{F}_q whose description follows from [24, 21, 22]. Let A denote the subring of $\mathcal{Q}(q)$ consisting of rational functions without pole at $q = 0$. The crystal lattice of \mathcal{F}_q is $L = \bigoplus_{\lambda \in \text{DP}_h} A|\lambda\rangle$, and the crystal basis of the \mathcal{Q} -vector space L/qL is $B = \{|\lambda\rangle \bmod qL, \lambda \in \text{DP}_h\}$. We shall write λ instead of $|\lambda\rangle \bmod qL$.

The Kashiwara operators \tilde{f}_i act on B in a simple way recorded on the crystal graph $\Gamma(\mathcal{F}_q)$. To describe this graph, one starts with the crystal graph $\Gamma(V_{\text{aff}})$ of V_{aff} . This is the graph with vertices $j \in \mathbb{Z}$, whose arrows labelled by $i \in \{0, 1, \dots, n\}$ are given, for $i \neq n$, by

$$j \xrightarrow{i} j + 1 \iff j \equiv n \pm i \pmod{h}$$

and for $i = n$ by

$$j \xrightarrow{n} j + 1 \iff j \equiv -1, 0 \pmod{h}.$$

Thus for $n = 2$ this graph is

$$\dots \xrightarrow{1} -1 \xrightarrow{2} 0 \xrightarrow{2} 1 \xrightarrow{1} 2 \xrightarrow{0} 3 \xrightarrow{1} 4 \xrightarrow{2} 5 \xrightarrow{2} 6 \xrightarrow{1} 7 \xrightarrow{0} \dots$$

The graph $\Gamma(\mathcal{F}_q)$ is obtained inductively from $\Gamma(V_{\text{aff}})$ using the following rules. Let $\lambda = (\lambda_1, \dots, \lambda_r) \in B$, and write $\lambda = (\lambda_1, \lambda^*)$ where $\lambda^* = (\lambda_2, \dots, \lambda_r)$. Then one has $\tilde{f}_i(0) = \delta_{in}(1)$, $\varphi_i(0) = \delta_{in}$, and

$$\tilde{f}_i \lambda = \begin{cases} (\tilde{f}_i \lambda_1, \lambda^*) & \text{if } \varepsilon_i(\lambda_1) \geq \varphi_i(\lambda^*) \\ (\lambda_1, \tilde{f}_i \lambda^*) & \text{if } \varepsilon_i(\lambda_1) < \varphi_i(\lambda^*). \end{cases}$$

Here, $\varepsilon_i(\lambda_1)$ means the distance in $\Gamma(V_{\text{aff}})$ from λ_1 to the origin of its i -string, and $\varphi_i(\lambda^*)$ means the distance in $\Gamma(\mathcal{F}_q)$ from λ^* to the end of its i -string.

Thus for $n = 1$ one computes successively the following 1-strings of $\Gamma(\mathcal{F}_q)$

$$\begin{aligned} (0) &\xrightarrow{1} (1) \\ (2) &= (2, 0) \xrightarrow{1} (2, 1) \xrightarrow{1} (3, 1) \xrightarrow{1} (4, 1) \\ (3, 2) &= (3, 2, 0) \xrightarrow{1} (3, 2, 1) \xrightarrow{1} (3, 3, 1) \xrightarrow{1} (4, 3, 1) \end{aligned}$$

from which one deduces that $\tilde{f}_1(3, 3, 1) = (4, 3, 1)$ and $\varphi_1(3, 3, 1) = 1$.

The first layers of the crystal $\Gamma(\mathcal{F}_q)$ for $n = 1$ are shown in figure 1. One can observe that the decomposition of $\Gamma(\mathcal{F}_q)$ into connected components reflects the decomposition (20) of \mathcal{F}_q into simple modules. More precisely, the connected components of $\Gamma(\mathcal{F}_q)$ are all isomorphic as coloured graphs to the component $\Gamma(\Lambda_n)$ containing the empty partition. Their highest vertices are the partitions ν whose parts are all divisible by h . This follows from the fact, easily deduced from the rules we have just explained, that if $\nu = h\mu = (h\mu_1, \dots, h\mu_r)$ is such a partition, then the map

$$\lambda \mapsto \lambda + \nu = (\lambda_1 + h\mu_1, \lambda_2 + h\mu_2, \dots) \tag{21}$$

is a bijection from $\Gamma(\Lambda_n)$ onto the connected component of $\Gamma(\mathcal{F}_q)$ containing ν , and this bijection commutes with the operators \tilde{e}_i and \tilde{f}_i . This implies that the vertices of $\Gamma(\Lambda_n)$ are the partitions $\lambda = (\lambda_1, \dots, \lambda_r, 0) \in \text{DP}_h$ such that for $i = 1, 2, \dots, r$, one has $\lambda_i - \lambda_{i+1} \leq h$ and $\lambda_i - \lambda_{i+1} < h$ if $\lambda_i \equiv 0 \pmod{h}$. We shall call a partition that satisfies these conditions h -regular. The set of h -regular partitions of m will be denoted by $\text{DPR}_h(m)$, and we shall write $\text{DPR}_h = \bigcup_m \text{DPR}_h(m)$.

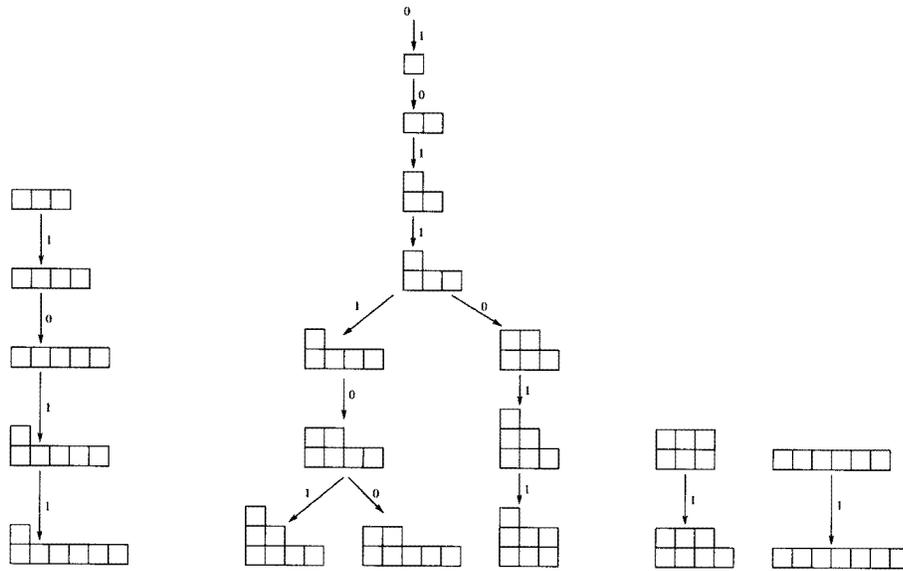


Figure 1. The graph $\Gamma(\mathcal{F}_q)$ for $A_2^{(2)}$ up to degree 7.

For example,

$$\text{DPR}_3(10) = \{(3331), (4321), (532), (541)\}.$$

4. The canonical basis of $V(\Lambda_n)$

In this section, we describe an algorithm for computing the canonical basis (global lower crystal basis) of the basic representation $V(\Lambda_n) = U_q(A_{2n}^{(2)})|0\rangle$ in terms of the natural basis $|\lambda\rangle$ of the q -Fock space. To characterize the canonical basis, we need the following notations

$$q_i = \begin{cases} q & \text{if } i = n \\ q^2 & \text{if } 1 \leq i < n \\ q^4 & \text{if } i = 0 \end{cases} \quad t_i = \begin{cases} q^{h_n} & \text{if } i = n \\ q^{2h_i} & \text{if } 1 \leq i < n \\ q^{4h_0} & \text{if } i = 0 \end{cases} \quad (22)$$

and

$$[k]_i = \frac{q_i^k - q_i^{-k}}{q_i - q_i^{-1}} \quad [k]_i! = [k]_i [k-1]_i \cdots [1]_i. \quad (23)$$

The q -divided powers of the Chevalley generators are defined by

$$e_i^{(k)} = \frac{e_i^k}{[k]_i!} \quad f_i^{(k)} = \frac{f_i^k}{[k]_i!}. \quad (24)$$

The canonical basis is defined in terms of an involution $v \mapsto \bar{v}$ of $V(\Lambda_n)$. Let $x \mapsto \bar{x}$ be the ring automorphism of $U_q(A_{2n}^{(2)})$ such that $\bar{q} = q^{-1}$, $\bar{q}^h = q^{-h}$ for h in the Cartan subalgebra of $A_{2n}^{(2)}$, and $\bar{e}_i = e_i$, $\bar{f}_i = f_i$. Then, for $v = x|0\rangle \in V(\Lambda_n)$, define $\bar{v} = \bar{x}|0\rangle$.

We denote by $U_{\bar{Q}}$ the sub- $\mathcal{Q}[q, q^{-1}]$ algebra of $U_q(A_{2n}^{(2)})$ generated by the $f_i^{(k)}$ and set $V_{\bar{Q}}(\Lambda_n) = U_{\bar{Q}}|0\rangle$. Then, as shown by Kashiwara [23], there exists a unique $\mathcal{Q}[q, q^{-1}]$ -basis $\{G(\mu), \mu \in \text{DPR}_h\}$ of $V_{\bar{Q}}(\Lambda_n)$, such that

Let $\mu^{(1)} < \mu^{(2)} < \dots < \mu^{(t)}$ be the set $\text{DPR}_h(m)$ sorted in lexicographic order, so that $A(\mu^{(i)}) = G(\mu^{(i)})$. Suppose that the expansion on the basis $|\lambda\rangle$ of $G(\mu^{(i+1)}), \dots, G(\mu^{(t)})$ has already been calculated. Then,

$$G(\mu^{(i)}) = A(\mu^{(i)}) - \gamma_{i+1}(q)G(\mu^{(i+1)}) - \dots - \gamma_i(q)G(\mu^{(i)}) \tag{27}$$

where the coefficients are determined by the conditions

$$\gamma_s(q^{-1}) = \gamma_s(q) \quad G(\mu^{(i)}) \equiv |\mu^{(i)}\rangle \pmod{qL}.$$

Thus, for $n = 1$, the first partition for which $A(\mu) \neq G(\mu)$ is $\mu = (3321)$ and

$$A(3321) = |3321\rangle + q|333\rangle + (q^2 - q^6)|432\rangle + (1 + 2q^2)|531\rangle + (q^2 + q^4)|54\rangle + (2q^2 + q^4)|621\rangle + 2q^3|63\rangle + (q^4 + q^6)|72\rangle + q^4|81\rangle + q^5|9\rangle. \tag{28}$$

Indeed, $A(3321) \equiv |3321\rangle + |531\rangle \pmod{qL}$. On the other hand, $A(531) = |531\rangle + q^2|54\rangle + q^2|621\rangle + q^3|63\rangle + q^6|72\rangle$ is equal to $G(531)$, and one finds by subtracting this from $A(3321)$ that

$$G(3321) = |3321\rangle + q|333\rangle + (q^2 - q^6)|432\rangle + 2q^2|531\rangle + q^4|54\rangle + (q^2 + q^4)|621\rangle + q^3|63\rangle + q^4|72\rangle + q^4|81\rangle + q^5|9\rangle. \tag{29}$$

Since $A(432) = |432\rangle + q^4|531\rangle + q^2|72\rangle + q^6|81\rangle$ satisfies (G1) and (G2), it has to be equal to $G(432)$, which completes the determination of the canonical basis for $m = 9$. For $m = 10$, the results are shown in table 1.

In the Fock-space representation of $A_{n-1}^{(1)}$, the weight of a basis vector $|\lambda\rangle$ is determined by the n core of the partition λ (and its degree) [4, 26]. There is a similar result of Nakajima and Yamada [33] for $A_{2n}^{(2)}$, in terms of the notion of \bar{h} core of a strict partition introduced by Morris [30] in the context of the modular representation theory of spin-symmetric groups.

One way to see this is to use a theorem of [31] according to which $\lambda, \mu \in \text{DP}(m)$ have the same \bar{h} core if and only if they have, for each i , the same number n_i of nodes of residue i . On the other hand, it follows from the implementation of the Chevalley generators that $|\lambda\rangle$ has $A_{2n}^{(2)}$ weight $\Lambda_n - \sum_{0 \leq i \leq n} n_i \alpha_i$, and the statement follows.

The definition of \bar{h} -cores can be extended to DP_h by deciding that if λ has repeated parts, its \bar{h} -core is equal to that of the partition obtained by removing those repeated parts. Then it is clear that if $|\lambda\rangle$ and $|\mu\rangle$ have the same $U_q(A_{2n}^{(2)})$ weight, the two partitions λ and μ have the same \bar{h} -core. It follows, since $G(\mu)$ is obviously a weight vector, that its expansion on the basis $|\lambda\rangle$ involves only partitions λ with the same \bar{h} -core as μ .

Table 1. The canonical basis for $n = 1$ and $m = 10$.

	(3331)	(4321)	(532)	(541)
(3331)	1	0	0	0
(4321)	$q - q^5$	1	0	0
(433)	q^2	q	0	0
(532)	0	0	1	0
(541)	$q + q^3$	$q^2 + q^4$	0	1
(631)	$2q^2$	q^3	0	q
(64)	q^4	0	0	q^3
(721)	$q^3 + q^5$	q^2	0	q^4
(73)	q^4	q^3	0	q^5
(82)	0	0	q^2	0
(91)	q^4	q^5	0	0
(10)	q^6	0	0	0

Summarizing the discussion, we have:

Theorem 4.1. For $\mu \in \text{DPR}_h(m)$, define $d_{\lambda\mu}(q)$ by $G(\mu) = \sum_{\lambda \in \text{DP}_h(m)} d_{\lambda\mu}(q) |\lambda\rangle$. Then,

- (i) $d_{\lambda\mu}(q) \in \mathbb{Z}[q]$,
- (ii) $d_{\lambda\mu}(q) = 0$ unless $\lambda \supseteq \mu$, and $d_{\mu\mu}(q) = 1$,
- (iii) $d_{\lambda\mu}(q) = 0$ unless λ and μ have the same \bar{h} -core.

5. The reduction $q = 1$

As observed by Kashiwara *et al* [24], to recover the classical Fock-space representation \mathcal{F} of $A_{2n}^{(2)}$, one has to introduce the inner product on \mathcal{F}_q for which the vectors $|\lambda\rangle$ are orthogonal and the adjoint operators of the Chevalley generators are

$$f_i^\dagger = q_i e_i t_i \quad e_i^\dagger = q_i f_i t_i^{-1} \quad t_i^\dagger = t_i. \tag{30}$$

It can be checked that, for $\lambda \in \text{DP}_h$,

$$\langle \lambda | \lambda \rangle = \prod_{k>0} \prod_{i=1}^{m_{kh}} (1 - (-q^2)^i) \tag{31}$$

where m_{kh} is the multiplicity of the part kh in λ .

Let \mathcal{F}_1 denote the $A_{2n}^{(2)}$ module obtained by specializing q to 1 as in [24]. This space is strictly larger than the classical Fock-space \mathcal{F} , since the dimension of its m th homogeneous component (in the principal gradation) is $|\text{DP}_h(m)|$ whereas that of \mathcal{F} is only $|\text{DP}(m)|$. Let $\mathcal{N} = \mathcal{F}_1^\perp$ denote the nullspace. It follows from (30) that \mathcal{N} is a $A_{2n}^{(2)}$ module, and from (31) that \mathcal{N} is the subspace of \mathcal{F}_1 spanned by the wedge products $|\lambda\rangle$ labelled by $\lambda \in \text{DP}_h - \text{DP}$. Therefore $\mathcal{F}_1/\mathcal{N}$ is a $A_{2n}^{(2)}$ module that can be identified with \mathcal{F} .

In this identification one has, for $\lambda = (\lambda_1, \dots, \lambda_r) \in \text{DP}$,

$$P_\lambda = 2^{\sum_{i=1}^r \lfloor (\lambda_i - 1)/h \rfloor} |\lambda\rangle. \tag{32}$$

The power of 2 comes from the fact that if $\lambda_i = kh$ for $k > 0$, and ν denotes the partition obtained from λ by replacing λ_i by $\nu_i = \lambda_i + 1$, then it follows from (1), (2) that $f_n P_\lambda$ contains P_ν with coefficient 1, while $f_n |\lambda\rangle$ contains $|\nu\rangle$ with coefficient 2 by (8). For later use we set

$$a_h(\lambda) = \sum_{i=1}^r \left\lfloor \frac{\lambda_i - 1}{h} \right\rfloor. \tag{33}$$

6. Modular representations of \widehat{S}_m

We refer the reader to [7] for an up-to-date review of the representation theory of the spin-symmetric groups and their combinatorics.

Let \widehat{S}_m be the spin-symmetric group as defined by Schur [35], that is, the group of order $2m!$ with generators z, s_1, \dots, s_{m-1} and relations $z^2 = 1, z s_i = s_i z, s_i^2 = z, (1 \leq i \leq m-1), s_i s_j = z s_j s_i$ ($|i - j| \geq 2$) and $(s_i s_{i+1})^3 = z$ ($1 \leq i \leq m-2$).

On an irreducible representation of \widehat{S}_m , the central element z has to act by $+1$ or by -1 . The representations for which $z = 1$ are actually linear representations of the symmetric group S_m , and those with $z = -1$, called spin representations, correspond to two-valued representations of S_m . The irreducible spin representations over a field of characteristic 0 are labelled, up to association, by strict partitions $\lambda \in \text{DP}(m)$. More precisely, let $\text{DP}_+(m)$ (resp. $\text{DP}_-(m)$) be the set of strict partitions of m having an even (resp. odd) number of even parts. Then, to each $\lambda \in \text{DP}_+(m)$ corresponds a self-associate irreducible spin character

$\langle \lambda \rangle$, and to each $\lambda \in \text{DP}_-(m)$ a pair of associate irreducible spin characters denoted by $\langle \lambda \rangle$ and $\langle \lambda \rangle'$.

According to Schur [35], the values $\langle \lambda \rangle(\rho)$ of the spin character $\langle \lambda \rangle$ on conjugacy classes of cycle-type $\rho = (1^{m_1}, 3^{m_3}, \dots)$ are given by the expansion of the symmetric function P_λ on the basis of power sums, namely

$$P_\lambda = \sum_{\rho} 2^{\lceil (\ell(\rho) - \ell(\lambda))/2 \rceil} \langle \lambda \rangle(\rho) \frac{P_\rho}{z_\rho} \quad (34)$$

where $z_\rho = \prod_j j^{m_j} m_j!$ and $\ell(\lambda)$ stands for the length of λ , that is the number of parts of λ .

For $\lambda \in \text{DP}(m)$, one introduces the self-associate spin character

$$\widehat{\langle \lambda \rangle} = \begin{cases} \langle \lambda \rangle & \text{if } \lambda \in \text{DP}_+(m) \\ \langle \lambda \rangle + \langle \lambda \rangle' & \text{if } \lambda \in \text{DP}_-(m). \end{cases} \quad (35)$$

The branching theorem for spin characters of Morris [30] implies that if $\widehat{\langle \lambda \rangle}$ gets identified with a weight vector of \mathcal{F} by setting

$$P_\lambda = 2^{\lfloor (m - \ell(\lambda))/2 \rfloor} \widehat{\langle \lambda \rangle} \quad (36)$$

then the b_∞ operator $f = \sum_{i \geq 0} f_i^\infty$ implements the induction of self-associate spin characters from \widehat{S}_m to \widehat{S}_{m+1} . Similarly, $e = e_0^\infty + 2 \sum_{i > 0} e_i^\infty$ implements the restriction from \widehat{S}_m to \widehat{S}_{m-1} . Thus, the Fock-space representation of b_∞ may be viewed as the sum $\mathcal{F} = \bigoplus_m \mathcal{C}(m)$ of additive groups generated by self-associate spin characters of \widehat{S}_m in characteristic 0. In this setting, the Chevalley generators of b_∞ act as refined induction and restriction operators.

Now, similarly to the case $A_{n-1}^{(1)}$, the reduction from b_∞ to $A_{2n}^{(2)}$ parallels the reduction modulo $p = h = 2n + 1$ of representations of \widehat{S}_m (from now on we assume that h is an odd prime).

More precisely, using (1), (2), (36), one sees immediately that the Chevalley generators f_i of $A_{2n}^{(2)}$ act on $\widehat{\langle \lambda \rangle}$ as the (r, \bar{r}) induction operators of Morris and Yaseen ($r = n + 1 - i$) [32]. Hence the vectors of degree m of $V(\Lambda_n) = U(A_{2n}^{(2)})^- |0\rangle$ can be identified with linear combinations of self-associate spin characters obtained by a sequence of (r, \bar{r}) inductions. It is known from modular representation theory that the maximal number of linearly independent self-associate projective spin characters of \widehat{S}_m in characteristic p is equal to the number of partitions of m into odd summands prime to p . Therefore the result follows at once from (5).

Theorem 6.1. The self-associate projective spin characters of \widehat{S}_m in characteristic p are linear combinations of characters obtained by a sequence of (r, \bar{r}) inductions.

This was proved by Bessenrodt *et al* [8] for $p = 3$ and Andrews *et al* [1] for $p = 5$, but the question remained open for $p \geq 7$ [7].

Moreover, the construction of section 4 gives an explicit basis for the space spanned by such characters. Denote by $\underline{A}(\mu)$ the column vector obtained from $A(\mu)$ by reduction $q = 1$ and expansion on the basis $\widehat{\langle \lambda \rangle}$. Then, $\underline{A}(\mu)$ is a projective character by (25) and $\{\underline{A}(\mu) \mid \mu \in \text{DPR}_p(m)\}$ is a basis of the \mathcal{Q} -vector space of self-associate projective spin characters of \widehat{S}_m in characteristic p .

These observations and the results of [26, 3, 15, 28] lead us to formulate a conjecture relating the global basis of $V(\Lambda_n)$ and the decomposition matrices for spin characters of the groups \widehat{S}_m .

Table 2. The decomposition matrix of \widehat{S}_{10} in characteristic 3.

	(3331)	(3331)'	(4321)	(4321)'	(532)	(541)	(541)'
(4321)	0	0	1	1	0	0	0
(532)	0	0	0	0	1	0	0
(532)'	0	0	0	0	1	0	0
(541)	1	1	1	1	0	0	1
(541)'	1	1	1	1	0	1	0
(631)	2	2	1	1	0	1	1
(631)'	2	2	1	1	0	1	1
(64)	1	1	0	0	0	1	1
(721)	1	1	0	1	0	0	1
(721)'	1	1	1	0	0	1	0
(73)	1	1	1	1	0	1	1
(82)	0	0	0	0	1	0	0
(91)	1	1	1	1	0	0	0
(10)	0	1	0	0	0	0	0
(10)'	1	0	0	0	0	0	0

Let $\mu \in \text{DPR}_p(m)$ and let $\underline{G}(\mu)$ stand for the image of the global basis $G(\mu)$ in $\mathcal{F} = \mathcal{F}_1/\mathcal{N}$, that is,

$$\underline{G}(\mu) = \sum_{\lambda \in \text{DP}(m)} 2^{b(\lambda) - a_p(\lambda)} d_{\lambda\mu}(1) \widehat{\lambda} \tag{37}$$

where $a_p(\lambda)$ is given by (33) and

$$b(\lambda) = \left\lfloor \frac{m - \ell(\lambda)}{2} \right\rfloor. \tag{38}$$

Then denote by $\underline{\underline{G}}(\mu)$ the vector obtained by factoring out the largest power of 2 dividing the coefficients of $\underline{G}(\mu)$ on the basis $\widehat{\lambda}$. For simplicity of notation, we shall identify $\underline{\underline{G}}(\mu)$ with the column vector of its coordinates on $\widehat{\lambda}$.

Finally, define the reduced decomposition matrix of \widehat{S}_m in characteristic p as the matrix obtained from the usual decomposition matrix for spin characters by adding up pairs of associate columns and expanding the column vectors so obtained on the basis $\widehat{\lambda}$. This is a matrix with $|\text{DP}(m)|$ rows and $|\text{DPR}_p(m)|$ columns. The definition is illustrated in tables 2 and 3. (Table 2 is taken from [32], except for the column labels which are ours and will be explained in the next section.)

Conjecture 6.2. (i) The set of column vectors of the reduced decomposition matrix of \widehat{S}_m in odd characteristic p such that $p^2 > m$ coincides with $\{\underline{\underline{G}}(\mu) | \mu \in \text{DPR}_p(m)\}$.

(ii) For $p^2 \leq m$, the reduced decomposition matrix of \widehat{S}_m is obtained by postmultiplying the matrix whose columns are $\underline{\underline{G}}(\mu)$ by a unitriangular matrix with non-negative entries.

Our conjecture has been checked on the numerical tables computed by Morris and Yaseen [32] ($p = 3$) and Yaseen [39] ($p = 5, 7, 11$). Thus, for $p = 3, m = 11$, the columns of the reduced decomposition matrix are

$$\underline{\underline{G}}(3332) \quad \underline{\underline{G}}(4331) + \underline{\underline{G}}(641) \quad \underline{\underline{G}}(5321) \quad \underline{\underline{G}}(542) \quad \underline{\underline{G}}(641).$$

Table 3. The reduced decomposition matrix of \widehat{S}_{10} in characteristic 3.

	(3331)	(4321)	(532)	(541)
$\widehat{(4321)}$	0	2	0	0
$\widehat{(532)}$	0	0	1	0
$\widehat{(541)}$	2	2	0	1
$\widehat{(631)}$	4	2	0	2
$\widehat{(64)}$	2	0	0	2
$\widehat{(721)}$	2	1	0	1
$\widehat{(73)}$	2	2	0	2
$\widehat{(82)}$	0	0	1	0
$\widehat{(91)}$	2	2	0	0
$\widehat{(10)}$	1	0	0	0

7. Labels for irreducible modular spin characters and partition identities

The labels for irreducible modular representations of symmetric groups form a subset of the ordinary labels [18]. It is therefore natural to look for a labelling scheme for irreducible modular spin representations of \widehat{S}_m using a subset of $\text{DP}(m)$. This was accomplished for $p = 3$ by Bessenrodt *et al* [8], who found that the Schur regular partitions of m form a convenient system of labels. These are the partitions $\lambda = (\lambda_1, \dots, \lambda_r)$ such that $\lambda_i - \lambda_{i+1} \geq 3$ for $i = 1, \dots, r - 1$, and $\lambda_i - \lambda_{i+1} > 3$ whenever $\lambda_i \equiv 0 \pmod{3}$.

In [8], it was also conjectured that for $p = 5$, the labels should be the partitions $\lambda = (\lambda_1, \dots, \lambda_r)$ satisfying the following conditions: (1) $\lambda_i > \lambda_{i+1}$ for $i \leq r - 1$, (2) $\lambda_i - \lambda_{i+2} \geq 5$ for $i \leq r - 2$, (3) $\lambda_i - \lambda_{i+2} > 5$ if $\lambda_i \equiv 0 \pmod{5}$ or if $\lambda_i + \lambda_{i+1} \equiv 0 \pmod{5}$ for $i \leq r - 2$, and (4) there are no subsequences of the following types (for some $j \geq 0$): $(5j+3, 5j+2)$, $(5j+6, 5j+4, 5j)$, $(5j+5, 5j+1, 5j-1)$, $(5j+6, 5j+5, 5j, 5j-1)$. This conjecture turned out to be equivalent to a q -series identity conjectured long ago by Andrews in the context of extensions of the Rogers–Ramanujan identities, and was eventually proved by Andrews *et al* [1]. The authors of [1] observed, however, that such a labelling scheme could not be extended to $p = 7, 11, 13$ (see also [7]).

In terms of canonical bases, the obstruction can be understood as follows. Assuming our conjecture and using the results of [8, 1], one can see that for $p = 3, 5$, the labels of [8] and [1] are exactly the partitions indexing the lowest non-zero entries in the columns of the matrices $D_m(q) = [d_{\lambda\mu}(q)]_{\lambda, \mu \vdash m}$. For example, in table 1, these are (10), (91), (82) and (73), which are indeed the Schur regular partitions of 10. The problem is that for $p \geq 7$, it can happen that two columns have the same partition indexing the lowest non-zero entry. For example, with $p = 7$ ($n = 3$) and $m = 21$, the two canonical basis vectors

$$G(75432) = |75432\rangle + q^2|76431\rangle + q|7752\rangle + q^3|7761\rangle + q^2|8643\rangle + (q^2 + q^4)|8652\rangle \\ + q^3|876\rangle + q^4|9543\rangle + (q^4 + q^6)|9651\rangle + q^5|975\rangle$$

and

$$G(654321) = |654321\rangle + q|75432\rangle + q|76431\rangle + q|76521\rangle + q^2|7743\rangle + q^2|7752\rangle \\ + q^2|7761\rangle + q^3|777\rangle + (q^3 + q^5)|8643\rangle + (q^3 + q^5)|8652\rangle \\ + (q^4 - q^8)|876\rangle + (q^3 + q^5)|9651\rangle + (q^4 + q^6)|975\rangle$$

have the same bottom partition (975) (compare [7], end of section 3).

On the other hand the partitions indexing the highest non-zero entries in the columns of $D_m(q)$ are the labels of the crystal graph (by theorem 4.1(ii)), so that they are necessarily

distinct. Therefore, we propose to use the set

$$\text{DPR}_p(m) = \{\lambda = (\lambda_1, \dots, \lambda_r) \vdash m \mid 0 < \lambda_i - \lambda_{i+1} \leq p \text{ if } \lambda_i \not\equiv 0 \pmod p, \\ 0 \leq \lambda_i - \lambda_{i+1} < p \text{ if } \lambda_i \equiv 0 \pmod p, (1 \leq i \leq r)\}$$

for labelling the irreducible spin representations of \widehat{S}_m in characteristic p . Indeed its definition is equally simple for all p . Moreover, because of theorem 4.1(iii), this labelling would be compatible with the p -block structure, which can be read on the \bar{p} -cores. Also, it is adapted to the calculation of the vectors $\underline{A}(\mu)$ which give an approximation to the reduced decomposition matrix.

Finally, we note that since DPR_p provides the right number of labels we have the following partition identity

$$\sum_{m \geq 0} |\text{DPR}_p(m)| t^m = \prod_{\substack{i \text{ odd} \\ i \not\equiv 0 \pmod p}} \frac{1}{1 - t^i} \tag{39}$$

which for $p = 3, 5$ is a counterpart to the Schur and Andrews–Bessenrodt–Olsson identities.

This happens to be a particular case of a theorem of Andrews and Olsson [2]. Namely, one gets (39) by taking $A = \{1, 2, 3, \dots, p - 1\}$ and $N = p$ in theorem 2 of [2]. A combinatorial proof of a refinement of the Andrews–Olsson partition identity has been given by Bessenrodt [6].

One can also get a direct proof of (39) without using representation theory by simply considering the bijections (21).

8. Discussion

We have used the level 1 q -deformed Fock spaces of Kashiwara *et al* [24] to compute the canonical basis of the basic representation of $U_q(A_{2n}^{(2)})$, and we have formulated a conjectural relation with the decomposition matrices of the spin symmetric groups in odd characteristic $p = 2n + 1$.

As in the case of $A_{n-1}^{(1)}$, it is reasonable to expect that in general, that is when $2n + 1$ is not required to be a prime, the canonical basis is related to a certain family of Hecke algebras at $(2n + 1)$ th roots of unity. A good candidate might be the Hecke–Clifford superalgebra introduced by Olshanski [34].

The case of $2n$ th roots of unity should then be related to the Fock-space representation of the affine Lie algebras of type $D_{n+1}^{(2)}$. In particular we believe that the fact used by Benson [5] and Bessenrodt–Olsson [9] that the 2-modular irreducible characters of \widehat{S}_m can be identified with the 2-modular irreducible characters of S_m corresponds in the realm of affine Lie algebras to the isomorphism $D_2^{(2)} \simeq A_1^{(1)}$.

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